MATH 2060 TUT02

Example Let $f : [0, \infty) \to \mathbb{R}$ be diff. on $(0, \infty)$.
If $\lim_{x\to\infty} f'(x) = \ell$, show that $\lim_{x\to\infty} \frac{f(x)}{x} = \ell$.

(The result follows immediately from L'Hopital'rule.
As a demonstration, we will prove it by MVT).

Ans: Let $\varepsilon > 0$.

Since $\lim_{x\to\infty} f'(x) = \ell$, $\exists\, c > 0$ s.t.

$$|f'(x) - \ell| < \varepsilon \quad \text{whenever } x > c.$$

For $x > c$, f is cts on $[c, x]$ and diff. on (c, x).

By MVT, $\exists\, \xi_x \in (c, x)$ s.t.

$$f(x) - f(c) = f'(\xi_x)(x - c)$$

$$\Rightarrow \quad \frac{f(x)}{x} - \ell = f'(\xi_x)\left(1 - \frac{c}{x}\right) - \ell + \frac{f(c)}{x}$$

$$= \underbrace{\left(f'(\xi_x) - \ell\right)}_{< \varepsilon}\underbrace{\left(1 - \frac{c}{x}\right)}_{< 1} \underbrace{- \ell \cdot \frac{c}{x} + \frac{f(c)}{x}}_{\text{small when } x \text{ large}}$$

Let $M := \max\{c(|\ell| + 1), |f(c)|\} > 0$.

Now if $x > M/\varepsilon$, then

$$\left|\frac{f(x)}{x} - \ell\right| \le |f'(\xi_x) - \ell|\left|1 - \frac{c}{x}\right| + \frac{|\ell| c}{x} + \frac{|f(c)|}{x}$$

$$< \varepsilon \cdot 1 + M(\varepsilon/M) + M(\varepsilon/M)$$

$$= 3\varepsilon$$

Therefore $\lim_{x\to\infty} \frac{f(x)}{x} = \ell$ //

Example Prove that the eqn $1+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}=0$ has one real root if n is odd and no real root if n is even.

Ans: Let $g(x):=1+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}$

Note g is cts and diff. on $\mathbb{R}$ with

$$g'(x)=1+x+x^2+\cdots+x^{n-1}=\begin{cases}\frac{x^n-1}{x-1} & \text{if } x\neq 1\\ n & \text{if } x=1\end{cases}$$

- Suppose n is odd,
  Then $g'(x)>0 \ \forall x\in\mathbb{R}$. $(x^n-1<0$ if $x<1$; $x^n-1>0$ if $x>1)$
  So g is strictly increasing on $\mathbb{R}$, and $g(x)=0$ has at most 1 real root.
  OTOH, since $\lim_{x\to-\infty} g(x)=-\infty$ (since n odd) and $\lim_{x\to\infty} g(x)=\infty$, Intermediate Value Thm implies that $g(x)=0$ has at least 1 real root.
  Hence $g(x)=0$ has exactly one real root

- Suppose n is even.
  Then $g'(-1)=0$.
  Moreover, if $x<-1$, then $x^n-1>0$, $x-1<0 \Rightarrow g'(x)<0$
  if $-1<x<0$, then $x^n-1<0$, $x-1<0 \Rightarrow g'(x)>0$
  if $x\geq 0$, then $g'(x)\geq 1>0$.

  So g has global min. at $x=-1$.
  Now, $\forall x\in\mathbb{R}$,

$$\begin{aligned} g(x)\geq g(-1) &= 1+(-1)+\frac{(-1)^2}{2}+\frac{(-1)^3}{3}+\cdots+\frac{(-1)^n}{n}\\ &=(1-1)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n-2}-\frac{1}{n-1}\right)+\frac{1}{n} \quad (n \text{ even})\\ &\geq \frac{1}{n}>0.\end{aligned}$$

  Hence $g(x)=0$ has no real root. //

Thm 6.3.1 (Preliminary Result)

Let
- $f, g : [a, b] \to \mathbb{R}$ $\quad (a < b)$
- $f(a) = g(a) = 0$
- $g(x) \neq 0 \quad \forall x \in (a, b)$.

If f and g are diff at a (1-sided) with $g'(a) \neq 0$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)}$ exists and $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

Thm 6.3.3 (L'Hopital's Rule I) $\left(\frac{0}{0}\right)$

Let
- $-\infty \leq a < b \leq \infty$
- f, g diff. on (a, b) (no assumption at end pts.)
- $g'(x) \neq 0$, $\forall x \in (a, b)$
- $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$

a) If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

b) If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \{\pm\infty\}$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

Thm 6.3.5 (L'Hopital's Rule II) $\left(\frac{\infty}{\infty}\right)$

Let
- $-\infty \leq a < b \leq \infty$
- f, g diff. on (a, b) (no assumption at end pts.)
- $g'(x) \neq 0$, $\forall x \in (a, b)$
- $\lim_{x\to a^+} g(x) = \pm\infty$

a) If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

b) If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \{\pm\infty\}$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

Example Let $f(x) := \begin{cases} x^2, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$, $g(x) := \sin x$, $x \in \mathbb{R}$.

Use Thm 6.3.1 to show that $\lim_{x \to 0} f(x)/g(x) = 0$

Explain why 6.3.3 cannot be used.

Ans: Check: $f(0) = g(0) = 0$, $g(x) \neq 0 \ \forall x \in (0, \pi)$,

f, g diff. at 0 with $f'(0) = 0$, $g'(0) = \cos 0 = 1 \neq 0$.

To see $f'(0) = 0$, note $\left| \frac{f(x) - f(0)}{x} \right| \leq |x| \quad \forall x \neq 0$

and apply squeeze thm

By Thm 6.3.1, $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = \frac{0}{1} = 0$.

However, Thm 6.3.3 cannot be used since $f'(x)$ DNE for $x \neq 0$

In fact f is NOT even cts for $x \neq 0$.

This can be seen readily by considering

rational $x_n \to x \quad \Rightarrow \quad f(x_n) = x_n^2 \to x^2$

irrational $y_n \to x \quad \Rightarrow \quad f(y_n) = 0$

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Example. Evaluate the following limits

a) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\operatorname{Arctan} x} \right)$ domain: $(0, \infty)$

Ans: Indeterminate form $\infty - \infty$.

Need to reduce to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ first.

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\operatorname{Arctan} x} \right)$$

$$= \lim_{x \to 0^+} \frac{\operatorname{Arctan} x - x}{x \cdot \operatorname{Arctan} x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1+x^2} - 1}{\operatorname{Arctan} x + \frac{x}{1+x^2}} \qquad \left(\begin{array}{l} (\operatorname{Arctan} x - x)' = \frac{1}{1+x^2} - 1 \text{ exists } \forall x > 0 \\ (x \cdot \operatorname{Arctan} x)' = \operatorname{Arctan} x + \frac{x}{1+x^2} \text{ exists } \neq 0 \ \forall x > 0 \end{array} \right)$$

$$= \lim_{x \to 0^+} \frac{-x^2}{(1+x^2)\operatorname{Arctan} x + x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0^+} \frac{-2x}{2 + 2x \operatorname{Arctan} x} \qquad \left(\begin{array}{c} (-x^2)' = -2x \text{ exists } \forall x > 0 \\ ((1+x^2)\operatorname{Arctan} x + x)' = 1 + 2x \operatorname{Arctan} x + 1 \\ \text{exists } \neq 0 \quad \forall x > 0. \end{array} \right)$$

L'Hopital's rule

$$= 0 \qquad (\text{limit exists, calculation justified})$$

Example. Evaluate the following limits

b) $\lim_{x\to 0^+} x^{\sin x}$ domain: $(0, \infty)$

Ans: Indeterminate form 0^0

Need to reduce to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ first.

Consider

$$\lim_{x\to 0^+} \ln\left(x^{\sin x}\right)$$

$$= \lim_{x\to 0^+} (\sin x)\ln x$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\csc x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} \qquad \left(\begin{array}{l}(\ln x)' = \frac{1}{x} \text{ exists } \forall x>0 \\ (\csc x)' = -\csc x \cot x \text{ exists } \neq 0 \ \forall x \in (0, \frac{\pi}{2})\end{array}\right)$$

L'Hopital's rule

$$= \lim_{x\to 0^+} \frac{\sin x}{x} \cdot (-\tan x)$$

$$= 1 \cdot 0 = 0$$

(limit exists, calculation justified

Finally, by continuity of exponential fcn $\exp: \mathbb{R} \to \mathbb{R}$,

$$\lim_{x\to 0^+} \exp\left(\ln\left(x^{\sin x}\right)\right) = \exp\left(\lim_{x\to 0^+} \ln\left(x^{\sin x}\right)\right) = e^0 = 1$$

i.e. $\lim_{x\to 0^+} x^{\sin x} = 1$

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Example Evaluate the limit

$$\lim_{x\to\infty} \frac{x + \sin x}{x - \sin x}$$

If we apply L'Hopital's rule "blindly",

then $$\lim_{x\to\infty} \frac{(x+\sin x)'}{(x-\sin x)'} = \lim_{x\to\infty} \frac{1+\cos x}{1-\cos x}$$

$$= \lim_{x\to\infty} \cot^2\left(\frac{x}{2}\right) \quad \text{DNE}$$

However, we cannot conclude that

$$\lim_{x\to\infty} \frac{x + \sin x}{x - \sin x} \quad \text{DNE also}$$

Ans: $$\lim_{x\to\infty} \frac{x + \sin x}{x - \sin x} = \lim_{x\to\infty} \frac{1 + \frac{\sin x}{x}}{1 - \frac{\sin x}{x}}$$

$$= \frac{1+0}{1-0} \quad (\text{squeeze thm})$$

$$= 1$$

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